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ALMOST ISOMETRIES ON THE UNIT BALL OF l_1

BY
GUN-MARIE LÖVBLOM
Department of Mathematics, University of Stockholm, Stockholm, Sweden; and The Royal Institute of Technology, Stockholm, Sweden

ABSTRACT
Let X and Y be Banach spaces. We consider the following problem: Given an almost isometry F from the unit ball of X into the unit ball of Y, does there exist an isometry near to F? It is shown that for $X = Y = l_1$ the answer is negative.

Let X, Y be Banach spaces and let $B_R(X)$ denote the ball with radius R and center 0 in X. We say that F is an ε-almost isometry of X if

$$(1-\varepsilon)\|x_1 - x_2\| \leqq \|F(x_1) - F(x_2)\| \leqq (1+\varepsilon)\|x_1 - x_2\| \qquad \text{for all } x \in X.$$

We are interested to know whether an ε-almost isometry of the unit ball is near to an isometry of the unit ball, or perhaps near to an isometry on a smaller ball. To be more precise we consider the following problem:

Given R, $0 < R < 1$, is there a function $\delta(\varepsilon)$ (depending on X, Y and R) such that $\delta(\varepsilon) \to 0$ when $\varepsilon \to 0$ and such that the following holds: Given ε-almost isometry $F : B_1(X) \to Y$ there is an isometry $G : B_R(X) \to Y$ such that $\|F(x) - G(x)\| \leqq \delta(\varepsilon)$ on $B_R(X)$?

For linear maps a positive answer has been obtained for $C(K)$-spaces by Benyamini [2] and for L_p-spaces by Alspach [1].

For non-linear maps a positive answer has been obtained for $C(K)$-spaces, K compact metric, by the author [3].

In the view of these positive results it may be somewhat surprising that if $X = Y = l_1$ the answer is negative for every R, $0 < R < 1$. More precisely we have the following theorem.

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THEOREM 1. *Let M be any number $\geqq 3$. For every $\varepsilon > 0$ there is an ε-almost isometry $F: B_1(l_1) \to B_1(l_1)$ such that for every isometry G defined on some subset of l_1 containing $B_{6/M}(l_1)$ into l_1 we have $\| G(x) - F(x) \| \geqq 1/M^2$ for some $x \in B_{3/M}(l_1)$.*

The problem is still open for $B_1(l_p)$, $p > 1$ and for $B_1(L_p)$, $p \geqq 1$. However, for $X = Y = (L_1(0, 1) \times \mathbf{R})_1$ the answer is negative and we have the following theorem.

THEOREM 2. *Let M be any number $\geqq 3$. For every $\varepsilon > 0$ there is an ε-almost isometry*

$$F: B_1((L_1(0, 1) \times \mathbf{R})_1) \to B_1((L_1(0, 1) \times \mathbf{R})_1)$$

such that for every isometry G defined on some subset of $(L_1(0, 1) \times \mathbf{R})_1$ containing $B_{6/M}((L_1(0, 1) \times \mathbf{R})_1)$ into $(L_1(0, 1) \times \mathbf{R})_1$ we have

$$\| G(x) - F(x) \| \geqq 1/2M^2 \qquad \text{for some } x \in B_{3/M}((L_1(0, 1) \times \mathbf{R})_1).$$

In $(L_1(0, 1) \times \mathbf{R})_1$ we have $\| (f, r) \| = \| f \|_{L_1} + | r |$.

In the proof of Theorem 1 we will use the following lemma. It is well-known, but for the sake of completeness we include a proof.

LEMMA 1. *Let M be a subset of l_1 and let $G: M \to l_1$ be an isometry with $G(0) = 0$. If $x, y, x + y \in M$ and x, y have disjoint supports then $G(x)$ and $G(y)$ have disjoint supports.*

PROOF. Let $G(x) = \Sigma\, a_i e_i$, $G(y) = \Sigma\, b_i e_i$, $G(x + y) = \Sigma\, c_i e_i$. Assume that a_N and b_N both are different from zero. Since

$$\Sigma | a_i - b_i | = \| G(x) - G(y) \| = \| x - y \| = \| x \| + \| y \| = \Sigma | a_i | + \Sigma | b_i |$$

we see that a_N and b_N have opposite signs.

We assume $a_N > 0$. We have

$$\| x \| + \| y \| = \| x + y \| = \| G(x + y) \| \leqq \| G(x + y) - G(x) \| + \| G(x) \| = \| y \| + \| x \|.$$

Since $a_N > 0$ we see that $c_N - a_N \geqq 0$ and hence $c_N > 0$.

On the other hand we also have

$$\| G(x + y) \| \leqq \| G(x + y) - G(y) \| + \| G(y) \| = \| x \| + \| y \|.$$

Since $b_N < 0$ we see that $c_N - b_N \leqq 0$ and hence $c_N < 0$, which gives a contradiction.

REMARK. An easy consequence of Lemma 1 is that if $G: B_R(l_1) \to B_R(l_1)$ is an isometry with $G(0) = 0$ and if $x, y \in B_{R/2}(l_1)$ have disjoint supports, then $G(x)$ and $G(y)$ have disjoint supports. However, we cannot conclude that G maps elements with disjoint supports onto elements with disjoint supports on bigger balls that $B_{R/2}(l_1)$. To show this we now give an example of an isometry $G: B_R(l_1) \to B_R(l_1)$ such that $G(re_1)$ and $G(re_2)$ do not have disjoint supports for $r > R/2$.

Let $a = \Sigma_{i=1}^{\infty} a_i e_i$ and define

$$G(a) = \sum_{i=1}^{\infty} (G(a))_i e_i$$

where

$$(G(a))_1 = \begin{cases} a_1 - R/2 & \text{if } a_1 > R/2, \\ -(a_2 - R/2) & \text{if } a_2 > R/2, \\ 0 & \text{otherwise;} \end{cases}$$

$$(G(a))_2 = \begin{cases} R/2 & \text{if } a_1 > R/2, \\ a_1 & \text{otherwise;} \end{cases}$$

$$(G(a))_3 = \begin{cases} R/2 & \text{if } a_2 > R/2, \\ a_2 & \text{otherwise;} \end{cases}$$

$$(G(a))_i = a_{i-1}, \qquad i \geqq 4.$$

Let $a = \Sigma a_i e_i$, $b = \Sigma b_i e_i \in B_R(l_1)$. To see that G is an isometry, we only need to check by symmetry the cases:

(a) $a_i, b_i \leqq R/2$, $i = 1, 2$;
(b) $a_1 > R/2$; $b_1, b_2 \leqq R/2$;
(c) $a_1, b_1 > R/2$;
(d) $a_1, b_2 > R/2$.

We check (d). Since $a_1, b_2 > R/2$ we have

$$\| F(a) - F(b) \| = \left| \left(a_1 - \frac{R}{2}\right) - \left(-\left(b_2 - \frac{R}{2}\right)\right)\right| + \left|\frac{R}{2} - b_1\right| + \left|\frac{R}{2} - a_2\right| + \sum_{i=3}^{\infty} | a_i - b_i |$$

$$= a_1 - b_1 + b_2 - a_2 + \sum_{i=3}^{\infty} | a_i - b_i |$$

$$= \| a - b \| .$$

We can similarly check (a), (b) and (c). We have, for every $r > R/2$,

$$(G(re_1))_1 = (r - R/2) = -(G(re_2))_1.$$

Proof of Theorem 1. We assume $\varepsilon = 1/n$ for some integer n and for $a \in l_1$ we define F_1 and F_2 by

$$F_1(a) = \begin{cases} a, & a_{n+1} \leqq 0 \\ a + \dfrac{a_{n+1}}{M}\left(\varepsilon \sum_{i=1}^{n} e_i - e_{n+1}\right), & a_{n+1} \geqq 0 \end{cases}$$

and $F_2(a) = \Sigma_{i=1}^{\infty} F_2(a_i e_i)$ where

$$F_2(a_i e_i) = \begin{cases} a_i e_{n+1+i}, & i \leqq n \text{ and } a_i < 2/M, \\ (a_i - 2/M)e_i + (2/M)e_{n+1+i}, & i \leqq n \text{ and } a_i \geqq 2/M, \\ a_{n+1} e_{n+1}, & i = n+1, \\ a_i e_{n+i}, & i > n+1. \end{cases}$$

Let $F = F_1 \circ F_2$. The strategy for proving that F is an almost isometry of $B(l_1)$ is that the supports of $F(\Sigma_{i=1}^{n} a_i e_i)$ and $F(a_{n+1}e_{n+1})$ will only overlap for those i for which $a_i > 2/M$. Since there are at most $M/2$ such a_i's if $\Sigma_{i=1}^{n} | a_i | \leqq 1$, we never get a big overlap.

To prove that F is far from an isometry we use Lemma 1. The overlap of the supports in the definition of $F(a_{n+1}e_{n+1})$ will make it impossible to be near to an isometry. We now give the details.

For $a \in l_1$ let $S_a = \{1, 2, \ldots, n\} \cap \operatorname{supp}(a)$. It is trivial to check that F_2 is an into isometry with $(F_2(a))_i \geqq 0$ for $1 \leqq i \leqq n$ and that, if $\| a \| \leqq 1$, then $\operatorname{card}(S_{F_2(a)}) \leqq M/2$.

Hence, to prove that F is an almost isometry, one only needs to prove:

CLAIM. If $\mathrm{card}(S_a)$, $\mathrm{card}(S_b) \leqq M/2$ and $a_i, b_i \geqq 0$ for $1 \leqq i \leqq n$, then we have

$$\| a - b \| \geqq \| F_1(a) - F_1(b) \| \geqq (1 - \varepsilon) \| a - b \| .$$

PROOF. We have to check three cases:

(i) If $a_{n+1}, b_{n+1} \leqq 0$ then clearly $\| F_1(a) - F_1(b) \| = \| a - b \|$.

(ii) If $a_{n+1} \geqq b_{n+1} \geqq 0$ then we have

$$\begin{aligned}\| F_1(a) - F_1(b) \| = & \sum_{\{1,\dots,n\}\setminus S_a \cup S_b} \varepsilon(a_{n+1} - b_{n+1})/M + (1 - 1/M)(a_{n+1} - b_{n+1}) \\ & + \sum_{S_a \setminus S_a \cap S_b} (a_i + \varepsilon(a_{n+1} - b_{n+1})/M) \\ & + \sum_{S_b \setminus S_a \cap S_b} | b_i - \varepsilon(a_{n+1} - b_{n+1})/M | \\ & + \sum_{S_a \cap S_b} | a_i - b_i + \varepsilon(a_{n+1} - b_{n+1})/M | + \sum_{n+2}^{\infty} | a_i - b_i | .\end{aligned}$$

By using the triangle inequality (on the third and fourth sum) and since $n\varepsilon = 1$, $\mathrm{card}(S_b) \leqq M/2$ we see that

$$\begin{aligned}\| a - b \| & \geqq \| F_1(a) - F_1(b) \| \\ & \geqq \| a - b \| - 2 \sum_{S_b} \varepsilon(a_{n+1} - b_{n+1}) \\ & = \| a - b \| - 2\varepsilon(\mathrm{card}(S_b))(a_{n+1} - b_{n+1}) \\ & \geqq (1 - \varepsilon) \| a - b \| .\end{aligned}$$

(iii) If $a_{n+1} \geqq 0 \geqq b_{n+1}$ then we have

$$\begin{aligned}\| F_1(a) - F_1(b) \| = & \sum_{\{1,\dots,n\}\setminus S_a \cup S_b} \varepsilon a_{n+1}/M + ((1 - 1/M)a_{n+1} - b_{n+1}) \\ & + \sum_{S_a \setminus S_a \cap S_b} (a_i + \varepsilon a_{n+1}/M) + \sum_{S_b \setminus S_a \cap S_b} | b_i - \varepsilon a_{n+1}/M | \\ & + \sum_{S_a \cap S_b} | a_i - b_i + \varepsilon a_{n+1}/M | + \sum_{n+2}^{\infty} | a_i - b_i | .\end{aligned}$$

By using the triangle inequality we see that

$$\| a - b \| \geqq \| F_1(a) - F_1(b) \| \geqq \| a - b \| - 2 \sum_{S_b} \varepsilon a_{n+1}/M$$

$$\geqq \| a-b \| - 2\varepsilon \operatorname{card}(S_b)(a_{n+1}-b_{n+1})/M \geqq \| a-b \| (1-\varepsilon)$$

since $\operatorname{card}(S_b) \leqq M/2$.

Thus we have proved that F is an ε-almost isometry.

Now, let G be any isometry defined on a subset of l_1 containing $B_{6/M}(l_1)$. If $G(0)=0$ then, by Lemma 1,

$$G\left(\frac{3}{M}e_i\right) \quad \text{and} \quad G\left(\frac{3}{M}e_j\right)$$

have disjoint supports if $i \neq j$. Let

$$S = \left\{ i \leqq n; \left(G\left(\frac{3}{M}e_i\right)\right)_i \neq 0 \right\}.$$

If $S = \{1, 2, \dots, n\}$ then

$$\left(G\left(\frac{3}{M}e_{n+1}\right)\right)_i = 0 \qquad \text{for all } i \leqq n.$$

Hence

$$\left\| F\left(\frac{3}{M}e_{n+1}\right) - G\left(\frac{3}{M}e_{n+1}\right) \right\| \geqq \sum_{i=1}^{n} \frac{3\varepsilon}{M^2} = \frac{3}{M^2}.$$

If $S \neq \{1, 2, \dots, n\}$, then for some $i \leqq n$ we have

$$\left(G\left(\frac{3}{M}e_i\right)\right)_i = 0$$

and therefore

$$\left\| G\left(\frac{3}{M}e_i\right) - F\left(\frac{3}{M}e_i\right) \right\| \geqq \frac{1}{M} \geqq \frac{3}{M^2}.$$

Now, if $0 \leqq \| G(0) \| \leqq 1/M^2$ set $H(x) = G(x) - G(0)$. Since H is an isometry with $H(0) = 0$, from the calculation above we get for some i, $i \leqq n+1$,

$$\frac{3}{M^2} \leqq \left\| H\left(\frac{3}{M}e_i\right) - F\left(\frac{3}{M}e_i\right) \right\| \leqq \left\| G\left(\frac{3}{M}e_i\right) - F\left(\frac{3}{M}e_i\right) \right\| + \| G(0) \|.$$

Thus

$$\left\| G\left(\frac{3}{M}e_i\right) - F\left(\frac{3}{M}e_i\right) \right\| \geqq \frac{2}{M^2}.$$

The remaining case $\| G(0) \| > 1/M^2$ is trivial.

This completes the proof of Theorem 1.

Let F be an ε-almost isometry defined on l_1. Then the following problem is open:

Given any $R > 0$, is there an isometry G defined on $B_R(l_1)$ and a $\delta = \delta(\varepsilon, R)$ such that $\delta \to 0$ when $\varepsilon \to 0$ and such that $\| G(x) - F(x) \| \leqq \delta$ on $B_R(l_1)$?

Since constructions of ε-almost isometries of the type used in the proof of Theorem 1 only work on bounded sets, the method in this paper seems to give no information on that question.

The proof of Theorem 2 is similar to the proof of Theorem 1, but for the sake of completeness we include a proof.

In the proof we will identify $(L_1(0, 1) \times \mathbf{R})_1$ with the space

$$X = \{ f \in L_1(0, 2) : f \equiv c \text{ on } [1, 2]\}$$

and use the following lemma.

Lemma 2. *Let M be a subset of $L_1(0, 2)$ and let $G : M \to L_1(0, 2)$ be an isometry with $G(0) = 0$. If $f, g, f + g \in M$ and $\{x : f(x) \neq 0\} \cap \{x : g(x) \neq 0\}$ has measure zero then $\{x : G(f)(x) \neq 0\} \cap \{x : G(g)(x) \neq 0\}$ has measure zero.*

The proof is similar to the proof of Lemma 1 and we omit it.

Proof of Theorem 2. We assume $\varepsilon = 1/n$ for some integer n.

Let $f \in B_1(X)$ with $f = a$ on $[1, 2]$ and define $F_1(f)$ by

$$F_1(f) = f \qquad \text{if } a \leqq 0$$

and if $a > 0$ we let

$$F_1(f)(x) = \begin{cases} f(x) + 2a/M & \text{on } [0, \frac{1}{2}), \\ f(x) & \text{on } [\frac{1}{2}, 1), \\ (1 - 1/M)a & \text{on } [1, 2]. \end{cases}$$

We define F_2 on $B_1(X)$ by

$$F_2(f)(x) = \begin{cases} \sup\{2(f(2x) - 2n/M), 0\} & \text{on } [0, \frac{1}{2}), \\ \inf\{2f(2x - 1), 4n/M\} & \text{on } [\frac{1}{2}, 1), \\ f(x) & \text{on } [1, 2]. \end{cases}$$

Now, let $F = F_1 \circ F_2$ and let $S_f = \{x \in [0, \frac{1}{2}]; f(x) > 0\}$. One can easily check that F_2 is an into isometry. Moreover, we have $F_2(f) \geqq 0$ on $[0, \frac{1}{2})$ and, since $\| f \| \leqq 1$, the measure of the set $S_{F_2(f)}$ is less than or equal to $M/4n = \varepsilon M/4$. Thus, to prove that F is an almost isometry one only needs to prove:

CLAIM. If $m(S_f)$, $m(S_g) \leqq \varepsilon M/4$ and $f, g \geqq 0$ on $[0, \frac{1}{2}]$ then we have

$$\| f - g \| \geqq \| F_1(f) - F_1(g) \| \geqq (1-\varepsilon) \| f - g \| .$$

PROOF. Let $f = a$, $g = b$ on $[1, 2]$. We have to check three cases:

(i) If $a, b \leqq 0$ then clearly $\| F_1(f) - F_1(g) \| = \| f - g \|$.

(ii) If $a > 0 \geqq b$ then we have

$$\begin{aligned} \| F_1(f) - F_1(g) \| &= \int_{[0,1/2]\setminus(S_f \cup S_g)} (2a/M)dx + \int_{S_f \setminus (S_f \cap S_g)} (f(x) + 2a/M)dx \\ &+ \int_{S_g \setminus (S_f \cap S_g)} |2a/M - g(x)|\,dx \\ &+ \int_{S_f \cap S_g} |f(x) - g(x) + 2a/M|\,dx \\ &+ \int_{[1/2,1]} |f(x) - g(x)|\,dx + \int_{[1,2]} ((1 - 1/M)a - b)dx. \end{aligned}$$

By using the triangle inequality and the fact that $m(S_g) \leqq \varepsilon M/4$ we see that

$$\begin{aligned} \| f - g \| \geqq \| F_1(f) - F_1(g) \| &\geqq \| f - g \| - 2 \int_{S_g} (2a/M)dx \\ &= \| f - g \| - 4am(S_g)/M \geqq \| f - g \| - \varepsilon a \geqq \| f - g \| (1 - \varepsilon). \end{aligned}$$

(iii) If $a \geqq b > 0$ then we have

$$\begin{aligned} \| F_1(f) - F_1(g) \| &= \int_{[0,1/2]\setminus(S_f \cup S_g)} 2(a-b)/M\,dx \\ &+ \int_{S_f \setminus (S_f \cap S_g)} (f(x) + 2(a-b)/M)dx \\ &+ \int_{S_g \setminus (S_f \cap S_g)} |g(x) - 2(a-b)/M|\,dx \\ &+ \int_{S_f \cap S_g} |f(x) - g(x) + 2(a-b)/M|\,dx \\ &+ \int_{[1/2,1]} |f(x) - g(x)|\,dx + \int_{[1,2]} (1 - 1/M)(a - b)dx. \end{aligned}$$

By using the triangle inequality we see that

$$\begin{aligned} \| f-g \| &\geqq \| F_1(f)-F_1(g) \| \\ &\geqq \| f-g \| -2\int_{S_g} 2(a-b)/M\,dx \\ &= \| f-g \| - 4(a-b)m(S_g)/M \\ &\geqq \| f-g \| - \varepsilon(a-b) \\ &\geqq \| f-g \| (1-\varepsilon). \end{aligned}$$

Thus we have proved that F is an ε-almost isometry.

Now, let $G: B_{6/M}(X) \to B(X)$ be an isometry and let $f_i \in B_{3/M}(X)$, $i = 1, 2, \ldots, n$ be defined by

$$f_i(x) = \begin{cases} 3n/M & \text{if } (i-1)/n \leqq x \leqq i/n \\ 0 & \text{otherwise} \end{cases}$$

and let

$$f_{n+1}(x) = \begin{cases} 0, & 0 \leqq x < 1 \\ 3/M, & 1 \leqq x \leqq 2. \end{cases}$$

Then $F(f_i)(x) = 2n/M$ on $[(i-1)/2n, i/2n]$ and $F(f_{n+1})(x) = 6/M^2$ on $[0, \frac{1}{2})$.

We first assume that $G(0) = 0$. Then by Lemma 2 we have

$$m(\{x : G(f_i)(x) \neq 0\} \cap \{x : G(f_j)(x) \neq 0\}) = 0 \qquad \text{if } i \neq j, \quad i, j \leqq n+1.$$

Set

$$A_i = \{x : G(f_i)(x) \neq 0\} \cap \left[\frac{i-1}{2n}, \frac{i}{2n}\right], \qquad i = 1, \ldots, n$$

and set

$$N = \{i \leqq n : m(A_i) \geqq 1/4n\}.$$

If $N \neq \{1, 2, \ldots, n\}$ then for some $i \leqq n$ we have $m(A_i) < 1/4n$ and hence we get

$$\begin{aligned} \| G(f_i) - F(f_i) \| &\geqq \int_{[(i-1)/2n, i/2n] \setminus A_i} |G(f_i)(x) - F(f_i)(x)|\,dx \\ &= \int_{[(i-1)/2n, i/2n] \setminus A_i} \frac{2n}{M}\,dx \end{aligned}$$

$$\geqq \frac{2n}{M} \cdot \frac{1}{4n} = \frac{1}{2M} .$$

Let

$$B_i = \left\{ x : G(f_{n+1})(x) \neq 0 \right\} \cap \left[\frac{i-1}{2n}, \frac{i}{2n} \right], \qquad i = 1, \dots, n.$$

If $N = \{1, 2, \dots, n\}$ then by Lemma 2 we have that $m(B_i) \leqq 1/4n$ for all $i \leqq n$. Therefore we get

$$\| G(f_{n+1}) - F(f_{n+1}) \| \geqq \int_{[0,1/2]} \left| G(f_{n+1})(x) - \frac{6}{M^2} \right| dx$$

$$\geqq \sum_{i=1}^{n} \int_{[(i-1)/2n, i/2n] \setminus B_i} \frac{6}{M^2} dx$$

$$\geqq n \cdot \frac{6}{M^2} \cdot \frac{1}{4n} = \frac{3}{2M^2} .$$

Now, if $0 < \| G(0) \| \leqq 1/2M^2$ set $H(f) = G(f) - G(0)$. Since $H : B_{6/M}(X) \to B(X)$ is an isometry with $H(0) = 0$, from the calculation above we have that

$$\| H(f) - F(f) \| \geqq \frac{3}{2M^2} \qquad \text{for some } f \in B_{3/M}(X).$$

Hence we get

$$\| G(f) - F(f) \| \geqq \frac{3}{2M^2} - \| G(0) \| \geqq \frac{1}{M^2} .$$

The remaining case $\| G(0) \| > 1/2M^2$ is trivial.

With this the proof of Theorem 2 is complete.

References

1. D. E. Alspach, *Small into isomorphisms on L_p spaces*, Illinois J. Math. **27** (1983), 300–314.
2. Y. Benyamini, *Small into isomorphisms between spaces of continuous functions*, Proc. Am. Math. Soc. **83** (1981), 479–485.
3. G.-M. Lövblom, *Isometries and almost isometries between spaces of continuous functions*, Isr. J. Math. **56** (1986), 143–159.